UKZN Matric Helpline Online Session: Arithmetic and Geometric Progressions

AP $T_n = a + (n-1)d$ $\qquad S_n = \frac{n}{2}[2a + (n-1)d]$

GP $T_n = ar^{n-1}$ $\qquad S_n = \frac{a(r^n - 1)}{r - 1};\ r \neq 1$ $\qquad S_\infty = \frac{a}{1-r};\ -1 < r < 1$

1. The sum of the first n terms of an arithmetic progression is given by

$$S_n = 5n^2 - 11n.$$

a. Determine the sum of the first ten terms.

b. Hence, determine the 10^{th} term.

a. $S_{10} = 5\times 10^2 - 11\times 10 = 390$

b. T_{10}?

$S_{10} = \cancel{T_1} + \cancel{T_2} + \ldots + \cancel{T_9} + T_{10}$

$S_9 = \cancel{T_1} + \cancel{T_2} + \ldots + \cancel{T_9}$

$T_{10} = S_{10} - S_9 = 390 - (5\times 9^2 - 11\times 9) = 84.$

c. Determine the arithmetic progression.

d. If the arithmetic progression consists of 20 terms, determine the sum of the last 10 terms.

$a = T_1$

$d = T_2 - T_1$

c. $T_n = a + (n-1)d$

$a = T_1 = S_1 = 5\times 1^2 - 11\times 1 = -6$

$S_2 = T_1 + T_2 = -6 + (5\times 2^2 - 11\times 2) = -2$

$\therefore T_1 + T_2 = -2 \qquad \therefore T_2 = -2 - (-6) = 4.$

$d = T_2 - T_1 = 4 - (-6) = 10$

$T_n = -6 + (n-1)\times 10 = -6 + 10n - 10 = 10n - 16.$

$-6, 4, 14, \ldots$

d. $S_{20} = \cancel{T_1} + \cancel{T_2} + \ldots + T_{10} + \underbrace{T_{11} + \ldots + T_{20}}$

$- \quad S_{10} = \cancel{T_1} + \cancel{T_2} + \ldots + T_{10}$

Sum of last 10 terms $= S_{20} - S_{10}$

$= 5\times 20^2 - 11\times 20 - 390$

$= 1780 - 390 = 1390$

e. If the arithmetic progression consists of 21 terms, determine the middle term.

$T_1 + T_2 + \dots + T_{10} + T_{11} + T_{12} + \dots + T_{21}$

$T_{11} = 10 \times 11 - 16 = 94.$

The middle term is 94.

$T_n = 10n - 16$

2. Consider the series: $5 + 13 + 21 + ... + (8n - 3)$.

a. Show that the sum to n terms is $4n^2 + n$.

$$5 + 13 + 21 + \dots + (8n-3)$$

$$8 \qquad 8$$

Here $d = 8$, $a = 5$

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] = \frac{n}{2}\left(2\times5 + (n-1)\times 8\right)$$

$$= \frac{n}{2}\left(10 + 8n - 8\right)$$

$$= \frac{n}{2}\left(8n + 2\right) = \frac{n}{2}\times 8n + \frac{n}{2}\times 2$$

$$= 4n^2 + n.$$

b. If the last four terms of the series are excluded, determine a formula for the sum of the remaining terms.

c. If the sum of the last four terms is 772, determine the number of terms in the series.

b. $S_n = \underbrace{T_1 + T_2 + \dots + T_{n-4}}_{S_{n-4}} + \underbrace{T_{n-3} + T_{n-2} + T_{n-1} + T_n}_{\text{last 4 terms.}}$

$$S_{n-4} = 4(n-4)^2 + (n-4)$$

$$= 4(n^2 - 8n + 16) + n - 4$$

$$= 4n^2 - 32n + 64 + n - 4$$

$$= 4n^2 - 31n + 60$$

$$S_{10} = \underbrace{T_1 + T_2 + T_3 + T_4 + T_5 + T_6}_{S_{10-4} = S_6} + \cancel{T_7} + \cancel{T_8} + \cancel{T_9} + \cancel{T_{10}}$$

$$S_6 = 4\times6^2 + 6 = 150.$$

Last four terms $= S_n - S_{n-4} = 772$

$$\therefore\ (\cancel{4n^2} + n) - (\cancel{4n^2} - 31n + 60) = 772$$

$$n - (-31n) - 60 = 772$$

$$32n = 772 + 60 = 832$$

$n = \frac{832}{32} = 26$

3. The first three terms of an arithmetic sequence are

$2k - 7, k + 8$ and $2k - 1$ where k is a constant from the set of Real numbers.

3.1 Determine the 15th term of the sequence.

3.2 Calculate the sum of the first thirty even numbers that are in the sequence.

$2k-7, \quad k+8, \quad 2k-1$ $\qquad T_{15}$?

$-k+15 \qquad k-9$

AP $\quad d = -k+15 = k-9$

$k+k = 15+9$

$2k = 24 \quad \therefore \; k = \frac{24}{2} = 12$

$d = -12+15 = 3, \quad a = 2\times12-7 = 17$

$T_n = a+(n-1)d = 17+(n-1)\times 3$

$= 3n + 14$

$T_{15} = 17+(15-1)\times3 = 59$

$17, 20, 23, 26, 29, 32, \ldots$

$20, 26, 32, \ldots$

$6 \qquad 6 \qquad a = 20, \quad d = 6$

$S_{30} = \frac{30}{2}\left[2\times20+(30-1)\times6\right]$

$= 3210$

4. A convergent geometric series consisting of only positive numbers has first term p, constant ratio r and n[th] term T_n such that $\sum_{n=3}^{\infty} T_n = \frac{1}{4}$.

4.1 If $T_1 + T_2 = 2$ then write down an expression for p in terms of r.

4.2 Determine the value(s) of r.

$T_n = pr^{n-1}$

4.1 $T_1 + T_2 = 2$,

$p + pr = 2$ $\quad \therefore (1+r)p = 2$

$\therefore \boxed{p = \frac{2}{1+r}}$

$T_1 = p$

$T_2 = pr^{2-1}$

$= pr$

4.2

$$S_\infty = \frac{a}{1-r} = \sum_{n=3}^{\infty} T_n = \frac{1}{4} = \underset{\underset{pr^2}{\shortparallel}}{T_3} + T_4 + \ldots$$

$$\therefore \frac{pr^2}{1-r} = \frac{1}{4}$$

$$\frac{\left(\frac{2}{1+r}\right) r^2}{1-r} = \frac{1}{4}$$

$$\therefore \frac{2r^2}{(1+r)(1-r)} = \frac{1}{4}$$

$$\therefore \frac{2r^2}{1-r^2} = \frac{1}{4}$$

$$\therefore 8r^2 = 1 - r^2$$

$$\therefore 9r^2 = 1$$

$$\therefore r^2 = \frac{1}{9} \quad \therefore r = \pm\frac{1}{3}$$

For positive values, $r = \frac{1}{3}$.